6.5 Methods of solving the D.E. at a glance

General form of the D.E $M(x, y)\,dx + N(x, y)\,dy = 0$

	Form of the D.E	Method of solving / solution
I	Variables separable form (Recapitulation)	
1.	$f(x)g(y)\,dx + F(x)G(y)\,dy = 0$	Divide by $g(y)F(x)$ and integrate.
2.	$\dfrac{dy}{dx} = f(ax+by+c)$	Put $ax+by+c = t$
3.	$\dfrac{dy}{dx} = \dfrac{(ax+by)+c}{k(ax+by)+c'}$	Put $ax+by = t$
II	**Homogeneous form**	
1.	$M(x, y)$ and $N(x, y)$ are homogeneous functions of the same degree with or without the involvement of terms with (y/x)	Write the D.E in the form $\dfrac{dy}{dx} = -\dfrac{M(x, y)}{N(x, y)}$ and put $y = vx$
	If homogeneous functions are involved with x/y	Write $\dfrac{dx}{dy} = -\dfrac{N(x, y)}{M(x, y)}$ and put $x = vy$
2.	$\dfrac{dy}{dx} = \dfrac{ax+by+c}{a'x+b'y+c'}$, $\dfrac{a}{a'} \neq \dfrac{b}{b'}$	Put $x = X+h$, $y = Y+k$ With proper choice of h and k the D.E reduces to a homogeneous D.E in X and Y. Put $Y = VX$ and solve.
III	**Exact form**	
1.	$\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ must be satisfied.	$\int M\,dx + \int N(y)\,dy = c$ is the solution.
2.	When $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$ then	Multiply the D.E with I.F to make it exact.
	(a) If $\dfrac{1}{N}\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right) = f(x)$	$e^{\int f(x)\,dx}$ is the I.F
	$\dfrac{1}{M}\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right) = g(y)$	$e^{-\int g(y)\,dy}$ is the I.F
	(b) $y f(xy)\,dx + x g(xy)\,dy = 0$	$\dfrac{1}{Mx - Ny}$ is the I.F
	(c) M and N involving terms of the form $x^a y^b$	$x^a y^b$ is the I.F where a and b are found such that $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

3.	Identifying the standard exact differentials and putting the D.E in the form $c_1 d\,[f_1(x, y)] + c_2 d\,[f_2(x, y)] + \cdots = 0$	$c_1 f_1(x, y) + c_2 f_2(x, y) + \cdots = c$ is the solution on integration
IV	**Linear form**	
1.	$\frac{dy}{dx} + Py = Q$ where P and Q are functions of x.	**Solution :** $y\,e^{\int P\,dx} = \int Q\,e^{\int P\,dx}\,dx + c$
2.	$\frac{dx}{dy} + Px = Q$ where P and Q are functions of y.	**Solution :** $x\,e^{\int P\,dy} = \int Q\,e^{\int P\,dy}\,dy + c$
3.	$f'(y)\frac{dy}{dx} + f(y)P = Q$ where $P = P(x)$ and $Q = Q(x)$	Put $f(y) = t$ and differentiate $w.r.t\ x$.
4.	$f'(x)\frac{dx}{dy} + f(x)P = Q$ where $P = P(y)$ and $Q = Q(y)$	Put $f(x) = t$ and differentiate $w.r.t\ y$.
5.	$\frac{dy}{dx} + Py = Qy^n$ where $P = P(x)$ and $Q = Q(x)$	Divide by y^n and put $y^{1-n} = t$ and diff. $w.r.t\ x$.
6.	$\frac{dx}{dy} + Px = Qx^n$ where $P = P(y)$ and $Q = Q(y)$	Divide by x^n and put $x^{1-n} = t$ and differentiate $w.r.t\ y$.

6.6 Type recognition - A retrospect

After having discussed several methods for solving a D.E an important aspect is *'type recognition'*. We have seen that a number of problems can be solved by more than one method. The reader should carefully take a note of several remarks made while discussing various types, **recognize the type and think of easier options if any before solving the problem in a particular method.**

A mixed set of problems are drawn from various examination papers and are just analysed for spotting the befitting method. It is left as an exercise for the reader to complete the problems that are not worked.

ANALYSIS OF PROBLEMS

Solve the following differential equations

1. $\frac{dy}{dx} - \frac{y}{x} = \sin\left(\frac{y}{x}\right)$

>> Observe the (y/x) terms. The D.E is homogeneous.

Put $y = vx$ and solve. *[Worked Problem - 8]*

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2. $\frac{dy}{dx} = \frac{7x - 3y - 7}{7y - 3x + 3}$

>> Observe that there is no common factor in the x, y terms.

Hence it is reducible to the homogeneous form. But before doing in this method check for exactness.

$$(7x - 3y - 7)\,dx - (7y - 3x + 7)\,dy = 0$$

$$\frac{\partial M}{\partial y} = -3, \quad \frac{\partial N}{\partial x} = 3 \quad \text{The equation is not exact.}$$

So it is inevitable to solve by using $x = X + h, \quad y = Y + k$ [*Worked Problem* - 20]

3. $(1 + y^2)\,dx + (\tan^{-1} y - x)\,dy = 0$

Observe that M is a function of y only and N contains x

Hence the equation is linear in x. Write it in respect of $\frac{dx}{dy}$ and solve.

[*Similar to worked Problem* - 64]

4. $\left(1 + e^{x/y}\right)dx + e^{x/y}\left(1 - \frac{x}{y}\right)dy = 0$

>> Observe (x/y) factors and that M and N are homogeneous functions of degree 0. The equation must be written in respect of $\frac{dx}{dy}$.

Solve by putting $x = vy$. Incidently this is also an exact equation.

[*Worked Problems* - 15, 35 *by both methods*]

5. $(x + \tan y)\,dy = \sin 2y\,dx$

Similar explanation as in Example - 3. The equation is linear in x.

[*Worked Problem* - 68]

6. $(y^2 + y + x)\,dy - y\,dx = 0$

>> Same explanation as of problem - 3.

$$\frac{dx}{dy} = \frac{y^2 + y + x}{y} \quad \text{or} \quad \frac{dx}{dy} - \frac{x}{y} = \frac{y^2 + y}{y}$$

ie., $\frac{dx}{dy} - \frac{x}{y} = (y + 1)$ is linear in x.

Aliter : The equation is neither separable nor homogeneous.

Let us try for exactness.

$$M = -y \quad N = (y^2 + y + x)$$

$$\frac{\partial M}{\partial y} = -1, \quad \frac{\partial N}{\partial x} = 1. \quad \text{The equation is not exact.}$$

Let us continue further.

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -2 \qquad \therefore \quad \frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{-2}{-y} = \frac{2}{y} = g(y)$$

$$\therefore \qquad e^{-\int g(y)\,dy} = e^{-\int \frac{2}{y}dy} = e^{-2\log y} = 1/y^2 \quad \text{is an I.F.}$$

Multiplying by $1/y^2$ we can solve as an exact equation.

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7. $\{x \tan(y/x) - y \sec^2(y/x)\}\,dx + x \sec^2(y/x)\,dy = 0$

>> Instantly one should recognize it to be a homogeneous equation as we have (y/x) terms homogeneous of degree 1. Write the equation in respect of $\frac{dy}{dx}$ and solve by putting $y = vx$. [*Worked Problem* - 10]

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8. $y\,dx - x\,dy = \sqrt{x^2 + y^2}\,dx$

>> Each of the terms are homogeneous function of degree 1.

This can be solved by putting $y = vx$. [*Similar Worked Problem* - 6]

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9. $\tan y \frac{dy}{dx} + \tan x = \cos y \cos^2 x$

>> This is looking like a linear equation in y. We should get rid off $\cos y$ in R.H.S.

Dividing by $\cos y$ we get $\sec y \ \tan y \frac{dy}{dx} + \sec y \ \tan x = \cos^2 x$

Put $\sec y = t$, the equation reduces to a linear equation in t.

[*Worked Problem* - 73]

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10. $\frac{dy}{dx} = \frac{2x - y + 1}{x + 2y - 3}$

No common factor in x, y terms and hence it is reducible to the homogeneous form. But before doing so, try for exactness.

$$(2x - y + 1)\,dx - (x + 2y - 3)\,dy = 0$$

$$M = 2x - y + 1 \quad \text{and} \quad N = -x - 2y + 3$$

$\frac{\partial M}{\partial y} = -1$ and $\frac{\partial N}{\partial x} = -1$. The equation is exact and we can write the solution instantly.

$$\int M\,dx + \int N(y)\,dy = c \qquad ie., \; x^2 - xy + x - y^2 + 3y = c.$$

Remark : *The thought exhibited in this Example and the gain in completing the problem easily be carefully noted.*

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11. $(1+x^2)\frac{dy}{dx} + 2xy = 6x^2$

>> Clearly in the standard form of a linear equation in y.
Dividing by $(1+x^2)$ we have,

$$\frac{dy}{dx} + \left(\frac{2x}{1+x^2}\right)y = \frac{6x^2}{1+x^2}$$

This can be solved by assuming the solution for the linear equation.

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12. $\frac{dy}{dx} - y\tan x = y^2 \sec x$

>> Clearly in the form which can be reduced to a linear equation. (*Bernoulli's form*). Divide by y^2 and put $1/y = t$.

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13. $(2 + 2x^2\sqrt{y})\,y\,dx + (x^2\sqrt{y} + 2)\,x\,dy = 0$

>> Separation of variables and homogeneous types can be ruled out at once. So, try exact.

$$\frac{\partial M}{\partial y} = 2 + 2x^2 \cdot \frac{3}{2}y^{1/2} = 2 + 3x^2\sqrt{y}$$

$$\frac{\partial N}{\partial x} = 3x^2\sqrt{y} + 2 \quad \text{The equation is exact.}$$

We can solve the exact equation easily. [*Worked Problem* - 36]

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14. $(x + 2y^3) = y\frac{dx}{dy}$

$$\frac{dx}{dy} = \frac{x+2y^3}{y} \quad \text{or} \quad \frac{dx}{dy} - \frac{x}{y} = 2y^2$$

>> This is a linear equation in x in the standard form which can be solved by assuming the solution for the linear equation.

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15. $x \sin(y/x)\, dy = [\, y \sin(y/x) - x \,]\, dx$

>> At once this can be recognized as a homogeneous equation.

We can solve by putting $y = vx$.

---

16. $\dfrac{dy}{dx} + \dfrac{y}{x}\log y = \dfrac{y}{x^2}$

>> It appears like a linear equation in y. So we have to get rid off y in the R.H.S. Dividing by y we get,

$$\frac{1}{y}\frac{dy}{dx} + \frac{1}{x}\log y = \frac{1}{x^2}$$

This can be solved by putting $\log y = t$.

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17. $\left(y^2 e^{xy^2} + 4x^3 \right) dx + \left[2xy\, e^{xy^2} - 3y^2 \right] dy = 0$

>> By ruling out the first two methods instantly we have to try for exactness. Infact this is an exact equation. [*Worked Problem* - 34]

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18. $\dfrac{dy}{dx}\left(x^2 y^3 + xy \right) = 1$

>> We have to write the equation in the form

$$\frac{dx}{dy} = x^2 y^3 + xy \quad \text{or} \quad \frac{dx}{dy} - xy = x^2 y^3$$

We have to divide by x^2 and later put $1/x = t$. [*Worked Problem* - 80]

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19. $x\dfrac{dy}{dx} + \dfrac{y^2}{x} = y$

>> This appears like a linear equation but we cannot put it in the proper form either dividing by x or by y. So let us simplify the problem.

$$x\frac{dy}{dx} = y - \frac{y^2}{x} \quad \text{or} \quad \frac{dy}{dx} = \frac{xy - y^2}{x^2}$$

This is a homogeneous equation and can be solved by putting $y = vx$

[*Worked Problem* - 7]

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20. $\frac{dy}{dx} = \frac{3y^2 + x^2}{2xy}$

>> At once we can recognize it to be a homogeneous equation and it can be solved by putting $y = vx$.

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21. $(x^2 + y^2 - a^2)\,x\,dx + (x^2 - y^2 - b^2)\,y\,dy = 0$

>> We can easily rule out the first two methods and hence try for exactness. Infact this is an exact equation.

---

22. $(y^3 - 3x^2 y)\,dx - (x^3 - 3xy^2)\,dy = 0$

>> It can be easily recognized as a homogeneous equation. But we can attempt for checking the exactness. Infact this is an exact equation also.
We can easily write the solution of the exact equation.

[*Worked Problem* 3 & 26 *in both the methods*]

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23. $x\frac{dy}{dx} + y = y^2 \log x$

>> It is appearing in the form of a linear equation and we have to divide by y^2 as well as by x.

We divide by $x y^2$ to obtain the equation in the form

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{xy} = \frac{\log x}{x}$$

Put $1/y = t$ to obtain a linear equation in t.

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24. $(y\cos x + \sin y + y)\,dx + (\sin x + x\cos y + x)\,dy = 0$

>> At once we can venture to try for exactness. Infact the equation is exact.

[*Worked Problem - 31*]

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25. $(x^2 y - 2xy^2)\,dx - (x^3 - 3x^2 y)\,dy = 0$

>> It can be recognized as a homogeneous equation at once. But it is not an exact equation. So we have to solve by putting $y = vx$.

---

26. $(2x - 10y^3)\,dy + y\,dx = 0$

>> This is linear in x and we put it in the form

$$\frac{dx}{dy} = \frac{10y^3 - 2x}{y} \quad \text{or} \quad \frac{dx}{dy} + \frac{2x}{y} = 10y^2$$

We can solve by using the associated standard solution.

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27. $(y - x + 1)\,dx - (y + x + 5)\,dy = 0$

>> Observe that there is no common factor in the x, y terms. So we can solve by reducing it to the homogeneous form by using the substitution $x = X + h$ and $y = Y + k$. This is inevitable as the equation is not exact.

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28. $y \sin 2x\,dx - (1 + y^2 + \cos^2 x)\,dy = 0$

At once we can rule out the first two methods and try for exactness. Infact the equation is exact.

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29. $xy^2\,dy - (x^3 + y^3)\,dx = 0$

It can be easily recognized as a homogeneous equation and is not an exact one. Hence we have to solve by putting $y = vx$.

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30. $x\,dy - y\,dx + \log y\,dy = 0$

We need to simplify the problem first *ie.,* $(x + \log y)\,dy = y\,dx$

$$\frac{dx}{dy} = \frac{x + \log y}{y} \quad \text{or} \quad \frac{dx}{dy} - \frac{x}{y} = \frac{\log y}{y}$$

This is a linear equation in x which can be solved by using the solution for the linear equation.
